

Inverse eigenvalue problem for discrete three-diagonal Sturm-Liouville operator and the continuum limit

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Abstract

In present article the self-contained derivation of eigenvalue inverse problem results is given by using a discrete approximation of the Schrödinger operator on a bounded interval as a finite three-diagonal symmetric Jacobi matrix. This derivation is more correct in comparison with previous works which used only single-diagonal matrix. It is demonstrated that inverse problem procedure is nothing else than well known Gram-Schmidt orthonormalization in Euclidean space for special vectors numbered by the space coordinate index. All the results of usual inverse problem with continuous coordinate are reobtained by employing a limiting procedure, including the Goursat problem – equation in partial derivatives for the solutions of the inversion integral equation.

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1 Introduction

There is a vast literature on the inverse scattering problem. Suffice it to mention the classical monographs [1-5], see also [6]. That theory has multiple applications and never stopped developing [7]. In parallel with its renovation, attempts to give a clear and obvious treatment were undertaken [8-10]. In these last papers that was done by using finite-difference approach which reduces the problem to solving relatively simple algebraic equations. Passage to the limit of the continuous variable allows one to obtain classical results of the inverse problem. Thus, the finite-difference version represents a valuable tool to reproduce all the results of the inversion procedure on a more accessible level of understanding. And this is not only of pedagogical interest. The aim of science is, among others, to supply a maximally compacted and clear knowledge free from superfluous and often obscure details.

The authors of [8-10] restricted their consideration to the finite-difference matrix Hamiltonian with potential coefficients only on the main diagonal of the operator. So,

there appears a disparity in numbers of interaction parameters and spectral data (see discussion below). As a result, we need either to impose some restrictions on spectral parameters or to introduce additional non-local potentials as was done in present article. That problem was not considered in [8-10], which led to an "erroneous" final result for the potential in the continuum limit: there must be additional factor 2 (missed in the articles mentioned) in front of derivative of the solution of inverse problem integral equation [see Eq. (37)]. This oversight was partially compensated in the paper [11] where the authors introduced non-local potentials which are needed for correct final results in the continuum limit. However, their method had several deficits, among them one can mention a rather "adjustable" character of the procedure suggested and none of their algorithms indicates the uniqueness.

At the same time, the inverse eigenvalue problem for the discrete analog of the Sturm-Liouville operator is now well developed. There is sufficiently large number of papers on that subject, see [12] where one can find most of the references. There are different variants of the inverse problem in the discrete approximation. All they deal with a matrix (finite or infinite) with several diagonals which may differ in number and which are recovered by given (spectral) parameters. For the Gel'fand-Levitan analog, the most known inversion variant, the central theorem is valid: Given the set of eigenvalues of a three-diagonal Jacobi matrix and the first components of the associated orthonormal eigenvectors, there exists a unique Jacobi matrix corresponding to these data, see e.g. [12, 13] and references therein. The usual proof of this theorem is performed using orthogonal polynomials [12].

Thus, we have the continuous and discrete variants where the inversion procedure is well established. However, considerably less was done to link them both. The problem is likely that the recovery procedures in discrete and continuum cases outwardly have little in common. In the continuum version, the inversion procedure (by Gel'fand-Levitan-Marchenko) is built as a transition from a certain known system (free motion, as a rule) to the system with known spectral data but with unknown potential to be reconstructed. The aim of the paper is to give such a derivation of the inverse eigenvalue problem from its discrete variant which would be free from the previous errors.

In the discrete variant, we have to develop, in a more explicit form, a structure similar to that in the continuum case. In doing so, we have to use a general criterion which would strongly specify whether our development is correct. The orthogonal polynomials' method gives such a hint. This is orthonormality relation (in a special spectral measure) which is valid for any system. As is well known, the Gram-Schmidt method is essential in constructing these polynomials. So the central idea of the present paper is to employ that method in order to reconstruct the set of eigenvectors orthonormalized in the initial spectral measure when the last is changed in a given way. We shall construct some "prototype" of the transformation procedure that realizes the reconstruction of the potential of regular Schrödinger operator in the continuum case.

We begin, in the second section, with several results consisting of some preliminary constructions which appear as intermediate steps in the course of the methods discussed. First of all, we give the discrete statement of the Sturm-Liouville problem on

a bounded interval with zero boundary conditions, which is equivalent to consideration of a three-diagonal symmetric Jacobi matrix. In the continuum limit these diagonals merge in a single diagonal (local potential). Then we pose the inverse eigenvalue problem in terms of eigenvalues and associated spectral weight factors, first components of the orthonormalized eigenvectors. Introducing so-called regular solutions admits of the explicit presentation of these parameters which serve as a spectral measure (of bounded support) entering in the Parseval relation for eigenvectors. That measure allows us to represent this equality as orthonormality condition for the same vectors but from another standpoint when the energies represent components and the discrete coordinate numbers the vectors. Next step is applying Gram-Schmidt technique to obtain the new orthonormal set of eigenvectors (in the sense mentioned) corresponding to the new measure. We shall see that procedure indeed reproduces prototypes for equations of the inverse problem in the continuum limit. The proof is given that such a orthogonalization is the only possible development. Next, we use the new eigenvectors to recover the potential coefficients on the three diagonals of the Jacobi matrix (discrete Sturm-Liouville operator) by using completeness relation for the new eigenvectors.

In the third section, we pass to the continuum limit. We demonstrate how all the discrete equations-prototypes go over into the classical equations of the inverse Sturm-Liouville problem: Gel'fand-Levitan equations, expression for the potential, classical Goursat problem, etc. That accomplishes our program.

2 Discrete version of inverse problem on finite interval

It is most easy to demonstrate the essence of the inverse eigenvalue problem for the example of finite-difference Schrödinger equation in the discrete variable $x_n, n \in \mathbb{Z}$ with the step Δ :

$$-\frac{\Psi(x_{n+1}, E) - 2\Psi(x_n, E) + \Psi(x_{n-1}, E)}{\Delta^2} + V(x_n)\Psi(x_n, E) + u(x_n)\Psi(x_{n+1}, E) + u(x_{n-1})\Psi(x_{n-1}, E) = E\Psi(x_n, E), \quad (1)$$

$V(x_n)$ and $u(x_n)$ are real, for this problem is reduced to linear algebraic equations. The first three terms in this equation represent finite-difference operator of the second derivative, i.e. kinetic energy. Note the existence, in the Schrödinger equation, of terms $u(x_n)$ corresponding to a "minimally non-local" interaction. We shall soon come back to them and their introduction will turn out justified.

Let us consider the bounded interval $[0, \pi]$ with finite number N of points inside: $x_0 = 0$; $x_{N+1} = \pi$, so that $\Delta = x_{n+1} - x_n = \pi/(N+1)$. Let us add the Eq. (1) by the Dirichlet boundary conditions:

$$\Psi(x_0, E) = \Psi(x_{N+1}, E) = 0. \quad (2)$$

These zero boundary conditions have the well-known physical interpretation that the movement of a particle is restricted by the infinitely tall walls at the points x_0 and x_{N+1} (the infinite rectangular potential well).

The spectrum of the problem (1), (2) is a ladder of discrete energy levels $\{E_\nu\}_{\nu=1}^N$ for bound states representing the unit vectors $\Psi_\nu(x_n) \equiv \Psi(x_n, E_\nu)$,

$$\sum_{n=0}^{N+1} \Delta \Psi_\mu(x_n) \Psi_\nu(x_n) = \delta_{\mu\nu}.$$

The Sturm-Liouville problem (1), (2) can be rewritten in a more visible form by using the symmetric tridiagonal (Jacobi) matrix $(N \times N)$

$$\begin{aligned} \hat{H} &= \hat{T} + \hat{J}, \\ \hat{T} &= \begin{pmatrix} 2/\Delta^2 & -1/\Delta^2 & 0 & 0 & \cdot & \cdot & \cdot \\ -1/\Delta^2 & 2/\Delta^2 & -1/\Delta^2 & 0 & \cdot & \cdot & \cdot \\ 0 & -1/\Delta^2 & 2/\Delta^2 & -1/\Delta^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & -1/\Delta^2 & 2/\Delta^2 \end{pmatrix}, \\ \hat{J} &= \begin{pmatrix} V(x_1) & u(x_1) & 0 & 0 & \cdot & \cdot & \cdot \\ u(x_1) & V(x_2) & u(x_2) & 0 & \cdot & \cdot & \cdot \\ 0 & u(x_2) & V(x_3) & u(x_3) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & u(x_{N-1}) & V(x_N) \end{pmatrix}, \end{aligned} \quad (3)$$

which acts on the vector-column $\Psi_\nu \in \mathbb{R}^N$:

$$\hat{H}\Psi_\nu = E_\nu\Psi_\nu,$$

$$\Psi_\nu = \begin{pmatrix} \Psi(x_1, E_\nu) \\ \Psi(x_2, E_\nu) \\ \cdot \\ \cdot \\ \Psi(x_N, E_\nu) \end{pmatrix}.$$

This explicit form elucidates why the $u(x_n)$ stands in front of $\Psi(x_{n+1}, E)$: that is because of the upper u-diagonal in \hat{J} which is one element shorter than the main V-diagonal and contains N-1 elements. The same coefficients form up the lower diagonal (thanks to the matrix symmetry). Let us specially note that homogeneous boundary conditions generally different from (2) would require some modification of the matrix

representation (3). In passing to the continuum limit $\Delta \rightarrow 0$, it will be impossible to distinguish the u- and V-diagonals, i.e., the resulting interaction will be simply the sum of limiting values for u's and V's. A paragraph later, we shall give the motivation for the appearance of the additional diagonals in the interaction matrix \hat{J} .

Besides the energy levels, let us introduce additional fundamental spectral parameters, namely, norming constants or spectral weight factors. By the definition, these are the coefficients c_ν of proportionality between the normalized eigenstates $\Psi_\nu(x_n)$ and the regular solutions $\varphi(x_n, E_\nu)$ at the eigenvalue energy, $\varphi(0, E) = 0, \varphi(x_1, E) = \Delta$ [i.e. the derivative is equal to 1], $\varphi_\nu(x_n) \equiv \varphi(x_n, E_\nu)$:

$$\Psi_\nu(x_n) = c_\nu \varphi_\nu(x_n). \quad (4)$$

The continuum analog of the regular solution satisfies $\varphi(0, E) = 0, \varphi'(x, E)|_{x=0} = 1$. The continuum generalization of the spectral weight factors introduced is obvious. In classical inverse Sturm-Liouville problem (continuous coordinate), it is well known that the double set of the spectral parameters $\{E_\nu, c_\nu\}$ uniquely specifies the potential.

Now we get to the core of our paper. Our task is to pose such a discrete version of the inverse Sturm-Liouville problem that, in passing to the continuum limit, as direct as possible reproduction of all the result of the continuum version is feasible. So, the problem within which it seems logical to work is posed as follows: Given the set $\{E_\nu, c_\nu\}$ with c_ν in (4), the potential matrix \hat{J} with coefficients $V(x_n)$ and $u(x_n)$ is recovered completely. In principle, a question may arise whether the three-diagonal Hamiltonian with the "non-local" u-coefficients is consistent with the uniqueness of the potential recovery from the set $\{E_\nu, c_\nu\}$. What does force the extra u 's? As was already mentioned in the introduction, any set $\{E_\nu, c_\nu\}$ can occur for a unique three-diagonal \hat{H} . Moreover, there is an extension of that result. It is the theorem by Gladwell and Willms [13] in which the statement was proved that a symmetric p-band matrix (a matrix with $2p + 1$ bands, p bands below the diagonal) may be uniquely constructed (apart from certain sign ambiguities) from its eigenvalues and the first p components of its normalized eigenvectors. Hence, once we know all E_ν 's and the first eigenvector components, by virtue of Eq. (4) these are Δc_ν in our problem, we can uniquely restore 1-band, i.e., three-diagonal Hamiltonian (3). That also accounts for the u 's.

Let us give an additional "half-heuristic" explanation of this fact. If we have only a local potential $V(x_n)$ with N values at N points, the number N of free parameters $\{V(x_n)\}_{n=1}^N$ equals exactly the number of eigenvalues E_ν . To the point, the corresponding inverse problem has not a complete solution. If we introduce both spectral parameters, E_ν and c_ν , while the single diagonal \hat{J} persists, we shall really face a problem of over-determination of the set $\{E_\nu, c_\nu\}$ that contains $2N-1$ free parameters. In fact, there are N levels E_ν and $N-1$ parameters c_ν by virtue of the relation $\sum_{\nu=1}^N c_\nu^2 = 1/\Delta^3$ that follows from (6) for $n = m = 1$, while $V(x_n)$ does only N ones. It is introduction of $N-1$ coefficients $u(x_n), n = 1, \dots, N-1$ into Eq. (1) [or additional diagonals in (3)] that ensures the equality of numbers of spectral data and interaction parameters. In the case of continuous coordinate, overfilling the set of spectral parameters reveals itself only in many-dimensional $D \geq 2$ problems, so we shall manage to restore one-dimensional

local potential by the complete spectral set [see (37)].

The functions $\varphi_\nu(x_n)$ can be considered as vectors in special Hilbert (Euclidean, to be precise) space, in which the coordinate x_n numbers the eigenvectors and energy index ν is only used to denote the ν th vector component. The inner product in that space is determined by the measure given by the spectral weight factors c_ν . In fact, the Parseval's completeness relation

$$\sum_{\nu=1}^N \Psi_\nu(x_m) \Psi_\nu(x_n) = \delta_{mn} / \Delta \quad (5)$$

can be rewritten using Eq.(4) as

$$\sum_{\nu=1}^N c_\nu^2 \varphi_\nu(x_m) \varphi_\nu(x_n) = \delta_{mn} / \Delta. \quad (6)$$

Let us consider this expression as an orthogonality relation for the vectors $\varphi_\nu(x_m)$ and $\varphi_\nu(x_n)$ (in the limit $\Delta \rightarrow 0$, the "numbers" $x_{m,n}$ of the vectors become continuous variable x). Here, the inner product is given by not simply a sum over energy index ν but a sum with a weight (measure) c_ν^2 .

Different potentials correspond to different weight factors determining the metrics of our "energy space" but the relation (6) holds true for any potential. In the classical variant, the inverse problem can be treated as a transition to the sought potential $\overset{\circ}{V}(x_n) \rightarrow V(x_n)$ from a certain "initial" (in what follows we shall use the symbol "o" to denote everything related to the initial system) potential $\overset{\circ}{V}(x_n)$, for which all the solutions $\overset{\circ}{\varphi}_\nu(x_n)$ and the whole spectral set $\{\overset{\circ}{E}_\nu, \overset{\circ}{c}_\nu\}$ are known, and the relation (6) is valid:

$$\sum_{\nu=1}^N \overset{\circ}{c}_\nu^2 \overset{\circ}{\varphi}_\nu(x_m) \overset{\circ}{\varphi}_\nu(x_n) = \delta_{mn} / \Delta. \quad (7)$$

All this gives us a hint for deriving new solutions corresponding to the given spectral set $\{E_\nu, c_\nu\}_{\nu=1}^N$. Although we do not know yet the sought potential matrix \hat{J} , we beforehand know that the regular solutions $\varphi_\nu(x_n)$ to these potentials must satisfy the orthogonality relation (6) with the new c_ν . We shall catch at this fact and use the orthogonality relation as a central criterion in finding new eigenvectors (solutions $\varphi_\nu(x_n)$). Changing the metrics of Euclidean space in replacing $\overset{\circ}{c}_\nu \rightarrow c_\nu$ results in that the "old" unit vectors $\overset{\circ}{\varphi}_\nu(x_n)$ are no longer orthogonal. So the idea is as follows: Once new unit vectors must satisfy Eq.(6), we could obtain them, e.g., orthogonalizing the $\overset{\circ}{\varphi}_\nu(x_n)$ with the new weight c_ν^2 by the Gram-Schmidt scheme. In other words, the new vectors obtained by that way and satisfying (6) with the weight multipliers c_ν^2 will be the solutions to the new potentials V 's and u 's. Indubitably, this makes sense only when the procedure really gives the desired vectors, i.e., it is unique (see the proof further on).

For simplicity, we shall at first think $\overset{\circ}{E}_\nu = E_\nu$. Let us recall this standard orthogonalization procedure for the example of two initially non-orthogonal (in sense of new weight function) vectors (i.e. when $N=2$) $\overset{\circ}{\varphi}_\nu(x_1), \overset{\circ}{\varphi}_\nu(x_2), \nu = 1, 2$. As a first unit vector $\varphi_\nu(x_1)$ of the new system, we take the unchanged unit vector $\overset{\circ}{\varphi}_\nu(x_1)$, and the second unit vector is constructed from the second unaltered one, only we have to subtract everything superfluous (parallel to $\overset{\circ}{\varphi}_\nu(x_1)$), for the orthogonality with the new measure:

$$\varphi_\nu(x_1) = \overset{\circ}{\varphi}_\nu(x_1); \quad \varphi_\nu(x_2) = \overset{\circ}{\varphi}_\nu(x_2) + \Delta K(x_2, x_1) \overset{\circ}{\varphi}_\nu(x_1).$$

The coefficient $K(x_2, x_1)$ is derived from the condition of orthogonality of the new vectors with the new weight c_ν :

$$\varphi_\nu(x_2) \perp \varphi_\nu(x_1) \equiv \overset{\circ}{\varphi}_\nu(x_1).$$

We have

$$\begin{aligned} & \sum_{\nu=1}^2 c_\nu^2 \overset{\circ}{\varphi}_\nu(x_1) \overset{\circ}{\varphi}_\nu(x_2) + \Delta \sum_{\nu=1}^2 c_\nu^2 K(x_2, x_1) \overset{\circ}{\varphi}_\nu^2(x_1) = 0 \implies \\ & K(x_2, x_1) + \sum_{\nu=1}^2 c_\nu^2 \overset{\circ}{\varphi}_\nu(x_1) \overset{\circ}{\varphi}_\nu(x_2) + \Delta K(x_2, x_1) \sum_{\nu=1}^2 (c_\nu^2 - \overset{\circ}{c}_\nu^2) \overset{\circ}{\varphi}_\nu^2(x_1) = 0, \end{aligned} \quad (8)$$

where we add and subtract the term $\overset{\circ}{c}_\nu^2$ from the multiplier c_ν^2 and, furthermore, use Eq. (7). We can rewrite the last equality in the form as follows (extremely simplified two-dimensional "prototype" of the inverse problem equation):

$$K(x_2, x_1) + Q(x_2, x_1) + \Delta K(x_2, x_1)Q(x_1, x_1) = 0, \quad (9)$$

where

$$\begin{aligned} Q(x_m, x_n) &= \sum_{\nu=1}^{N=2} c_\nu^2 \overset{\circ}{\varphi}_\nu(x_m) \overset{\circ}{\varphi}_\nu(x_n) - \\ & \sum_{\mu=1}^{N=2} \overset{\circ}{c}_\mu^2 \overset{\circ}{\varphi}_\mu(x_m) \overset{\circ}{\varphi}_\mu(x_n); \quad m, n = 1, 2. \end{aligned} \quad (10)$$

In general case of N -dimensional Euclidean space we shall follow the same scheme. In doing so, it is possible to take into account the case when the levels change: $\overset{\circ}{E}_\nu \neq E_\nu$. We must orthogonalize N vectors by the measure c_ν^2 : $\overset{\circ}{\varphi}(x_m, E_\nu)$, ($m = 1, 2, \dots, N, \nu = 1, 2, \dots, N$). Consequently, we have for the new solutions

$$\varphi(x_m, E_\nu) = \overset{\circ}{\varphi}(x_m, E_\nu) + \sum_{n=1}^{m-1} \Delta K(x_m, x_n) \overset{\circ}{\varphi}(x_n, E_\nu), \quad (11)$$

where the coefficients K [the kernel of the transformation operator (11)] follow from the conditions of the orthogonality of new vectors (by measure c_ν^2) $\varphi(x_m, E_\nu)$ ($m = 1, 2, \dots, N$):

$$\varphi(x_{m>n}, E_\nu) \perp \varphi(x_n, E_\nu),$$

which lead to the system of algebraic equations for K – discrete analog of central equations of the inverse problem:

$$K(x_m, x_n) + Q(x_m, x_n) + \sum_{p=1}^{m-1} \Delta K(x_m, x_p) Q(x_p, x_n) = 0, \quad m > n, \quad (12)$$

where $Q(x_n, x_m)$ is determined as in (10), only the values m and n are no longer restricted by 1 and 2, and the indices μ and ν number solutions at initial and shifted energy levels, respectively.

$$Q(x_m, x_n) = \sum_{\nu=1}^N c_\nu^2 \overset{\circ}{\varphi}(x_m, E_\nu) \overset{\circ}{\varphi}(x_n, E_\nu) - \sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_\mu). \quad (13)$$

Let us note that the form (11) ensures the desired boundary condition for the regular solution: $\varphi(x_0, E_\nu) = 0$; $\varphi(x_1, E_\nu) = \Delta$. The system Eq. (12) of the recurrence computation of the K 's provides them uniquely. We can formally introduce the diagonal terms $K(x_n, x_n)$ (bearing no relation to $\varphi(x, E_\nu)$) such as $K(x_{n+1}, x_n) - K(x_n, x_n) \sim O(\Delta)$ which will be useful in what follows.

When the first $m+1$ unit vectors $\overset{\circ}{\varphi}(x_i, E_\nu)$, $i = 1, \dots, m+1$ are orthogonalized, this corresponds to an intermediate submatrix-block transformation of the initial Jacobi-like operator (3) so that

$$\hat{J} = \begin{pmatrix} \hat{J}_m & \mathbf{0} \\ \mathbf{0} & \overset{\circ}{J}_{N-m} \end{pmatrix},$$

$$\hat{J}_m = \begin{pmatrix} V(x_1) & u(x_1) & 0 & 0 & \cdot & \cdot & \cdot \\ u(x_1) & V(x_2) & u(x_2) & 0 & \cdot & \cdot & \cdot \\ 0 & u(x_2) & V(x_3) & u(x_3) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & u(x_{m-1}) & V(x_m) & \overset{\circ}{u}(x_m) \end{pmatrix},$$

$$\overset{\circ}{J}_{N-m} = \begin{pmatrix} \overset{\circ}{u}(x_m) & \overset{\circ}{V}(x_{m+1}) & \overset{\circ}{u}(x_{m+1}) & 0 & \cdot & \cdot & \cdot \\ 0 & \overset{\circ}{u}(x_{m+1}) & \overset{\circ}{V}(x_{m+2}) & \overset{\circ}{u}(x_{m+2}) & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \overset{\circ}{u}(x_{N-1}) & \overset{\circ}{V}(x_N) \end{pmatrix}, \quad (14)$$

where the symbols $\mathbf{0}$ in the top-right and bottom-left corners of the \hat{J} -matrix denote zero $(m \times N - m - 1)$ and $(N - m \times m - 1)$ matrices, respectively. The submatrix \hat{J}_m is formed up by the perturbed coefficients while the \hat{J}_{N-m} is not affected yet by the transformation associated with the reorthogonalization. Note that the last row of the submatrix \hat{J}_m contains only two transformed elements, $u(x_{m-1})$ and $V(x_m)$, apart from the $\overset{\circ}{u}(x_m)$. This is because the $(m \times m)$ *quadratic* submatrix was transformed only which contains, in its last row, two non-zero elements mentioned. The coefficients of that intermediate transformation block may be found from formulas (20) and (21) where one should substitute $\overset{\circ}{u}(x_m)$ for $u(x_m)$. See also the formulas (20) and (21) and the subsequent discussion.

Now we are ready to give the proof that the above procedure is unique. Let us carry out it by induction. Suppose that the desired transformed vectors are uniquely given by the equation (11) for $m \leq \bar{N}$, $\bar{N} < N$ for a certain \bar{N} being the integer. For $\bar{N} = 1$, this is verified trivially. Let us show that the formula (11) holds true at the point $x_{\bar{N}+1}$. Indeed, the $\varphi(x_{\bar{N}+1}, E_\nu)$, being orthogonal to all $\varphi(x_m, E_\nu)$, $m \leq \bar{N}$, can be sought, in principle, as a combination of the initial $\overset{\circ}{\varphi}(x_n, E_\nu)$ for all n . The coefficients of such a hypothetical combination (there are N pieces of them in all) have to be determined from the condition of orthogonality of $\varphi(x_{\bar{N}+1}, E_\nu)$ to both $\varphi(x_m, E_\nu)$, $m = 1, \dots, \bar{N}$ and certain $N - \bar{N}$ unknown vectors from the new orthogonal set (6). For ascertaining what we shall do now, we involve the Schrödinger equation (1), which is a recurrence procedure of the step-by-step computation of the $\varphi(x_n, E_\nu)$. Specifically, we mean the block operator $\hat{J}_{\bar{N}}$ in (14), last row:

$$-\frac{\varphi(x_{\bar{N}+1}, E_\nu) - 2\varphi(x_{\bar{N}}, E_\nu) + \varphi(x_{\bar{N}-1}, E_\nu)}{\Delta^2} + V(x_{\bar{N}})\varphi(x_{\bar{N}}, E_\nu) + \overset{\circ}{u}(x_{\bar{N}})\varphi(x_{\bar{N}+1}, E_\nu) + u(x_{\bar{N}-1})\varphi(x_{\bar{N}-1}, E_\nu) = E_\nu\varphi(x_{\bar{N}}, E_\nu).$$

It is seen that $\overset{\circ}{u}(x_{\bar{N}})\varphi(x_{\bar{N}+1}, E_\nu)$ is a linear combination of $u(x_{\bar{N}-1})\varphi(x_{\bar{N}-1}, E_\nu)$, $V(x_{\bar{N}})\varphi(x_{\bar{N}}, E_\nu)$ and $E_\nu\varphi(x_{\bar{N}}, E_\nu)$. Consequently, by the assumptions of the validity of (11) for $m \leq \bar{N}$, $\overset{\circ}{u}(x_{\bar{N}})\varphi(x_{\bar{N}+1}, E_\nu)$ can only be represented through $V(x_{\bar{N}})K(x_{\bar{N}}, x_m)\overset{\circ}{\varphi}(x_m, E_\nu)$, $m = 1, \dots, \bar{N} - 1$; $V(x_{\bar{N}})\overset{\circ}{\varphi}(x_{\bar{N}}, E_\nu)$; $u(x_{\bar{N}-1})K(x_{\bar{N}-1}, x_m)\overset{\circ}{\varphi}(x_m, E_\nu)$, $m = 1, \dots, \bar{N} - 2$; $u(x_{\bar{N}-1})\overset{\circ}{\varphi}(x_{\bar{N}-1}, E_\nu)$; $K(x_{\bar{N}-1}, x_m)E_\nu\overset{\circ}{\varphi}(x_m, E_\nu)$, $m = 1, \dots, \bar{N} - 1$; and $E_\nu\overset{\circ}{\varphi}(x_{\bar{N}}, E_\nu)$. For the last term, we find from the non-perturbed Schrödinger equation that $E_\nu\overset{\circ}{\varphi}(x_{\bar{N}}, E_\nu)$ is expressed through $\overset{\circ}{u}(x_{\bar{N}})\overset{\circ}{\varphi}(x_{\bar{N}+1}, E_\nu)$, $\overset{\circ}{V}(x_{\bar{N}})\overset{\circ}{\varphi}(x_{\bar{N}}, E_\nu)$, $\overset{\circ}{u}(x_{\bar{N}-1})\overset{\circ}{\varphi}(x_{\bar{N}-1}, E_\nu)$ [and similarly for other $E_\nu\overset{\circ}{\varphi}(x_m, E_\nu)$] and, finally, the $\varphi(x_{\bar{N}+1}, E_\nu)$ must be sought as a linear combination of $\overset{\circ}{\varphi}(x_m, E_\nu)$, $m = 1, \dots, \bar{N} + 1$. In other words, $\varphi(x_{\bar{N}+1}, E_\nu) \in \text{span}\{\overset{\circ}{\varphi}(x_m, E_\nu)\}_{m=1}^{\bar{N}+1}$, besides that $\varphi(x_{\bar{N}+1}, E_\nu) \perp \text{span}\{\overset{\circ}{\varphi}(x_m, E_\nu)\}_{m=1}^{\bar{N}} = \text{span}\{\varphi(x_m, E_\nu)\}_{m=1}^{\bar{N}}$. It is well known that the Gram-Schmidt orthogonalization enables a unique solution satisfying these two conditions. Note that $(\bar{N}+1)$ th term $\overset{\circ}{u}(x_{\bar{N}})\varphi(x_{\bar{N}+1}, E_\nu)$ is a combination of the summand $\overset{\circ}{u}(x_{\bar{N}})\overset{\circ}{\varphi}(x_{\bar{N}+1}, E_\nu)$

(with the same coefficient) and other terms with $m \leq \bar{N}$. In other words, in the decomposition of $\varphi(x_{\bar{N}+1}, E_\nu)$, we have the term $\overset{\circ}{\varphi}(x_{\bar{N}+1}, E_\nu)$ with the unit coefficient. Consequently, $\varphi(x_{\bar{N}+1}, E_\nu)$ is represented in the form (11) again (QED).

It should be noted that the formula (11) is also valid for solutions at energies E lying between the levels E_ν where the regular solutions, being the Cauchy problem solutions, are defined (though non-physical). Do not confuse "running" energy values at which the solutions $\varphi(x_m, E)$ are defined with the energies occurring in the inverse problem equations (13). Indeed, let us decompose $\varphi(x_m, E)$ into the complete set of the solutions $\varphi(x_m, E_\nu)$ (in sense of the usual inner product $\sum_{m=1}^N \Delta c_\nu c_\mu \varphi(x_m, E_\nu) \varphi(x_m, E_\mu)$):

$$\varphi(x_m, E) = \sum_{\nu=1}^N \xi(E, E_\nu) \varphi(x_m, E_\nu); \quad \xi(E, E_\nu) = \sum_{m=1}^N \Delta c_\nu \varphi(x_m, E_\nu) \varphi(x_m, E). \quad (15)$$

Since the $\varphi(x_m, E_\nu)$'s are expressed, in accordance to (11), through the unperturbed $\overset{\circ}{\varphi}(x_m, E_\nu)$ then we shall expand them, too, in a complete set of the old solutions $\overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu)$:

$$\overset{\circ}{\varphi}(x_m, E_\nu) = \sum_{\mu=1}^N \zeta(E_\nu, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu); \quad \zeta(E_\nu, \overset{\circ}{E}_\mu) = \sum_{m=1}^N \Delta \overset{\circ}{c}_\mu \overset{\circ}{\varphi}(x_m, E_\nu) \overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu). \quad (16)$$

Combining (11), (15) and (16), we get the following expression for the new solutions at arbitrary E :

$$\begin{aligned} \varphi(x_m, E) &= \sum_{\mu, \nu} A(E, E_\nu, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu) + \sum_{\mu, \nu} \sum_{n=1}^{m-1} \Delta K(x_m, x_n) \\ &\times A(E, E_\nu, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_\mu); \quad A(E, E_\nu, \overset{\circ}{E}_\mu) = \xi(E, E_\nu) \zeta(E_\nu, \overset{\circ}{E}_\mu). \end{aligned} \quad (17)$$

In the limit when the new and old spectral parameters coincide, K vanishes and, hence, $\varphi(x_m, E)$ turn into unperturbed solution

$$\overset{\circ}{\varphi}(x_m, E) = \sum_{\mu, \nu} A(E, E_\nu, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_m, \overset{\circ}{E}_\mu).$$

Substituting this expression in (17), we get:

$$\varphi(x_m, E) = \overset{\circ}{\varphi}(x_m, E) + \sum_{n=1}^{m-1} \Delta K(x_m, x_n) \overset{\circ}{\varphi}(x_n, E). \quad (18)$$

Let us stress here that K is independent of energy E . The formulas (18), (12) and (13) give the expression for K in the form of sum of products of the old solutions and

transformed ones:

$$K(x_m, x_n) = - \sum_{\nu}^N c_{\nu}^2 \varphi(x_m, E_{\nu}) \overset{\circ}{\varphi}(x_n, E_{\nu}) + \sum_{\mu}^N \overset{\circ}{c}_{\mu}^2 \varphi(x_m, \overset{\circ}{E}_{\mu}) \overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_{\mu}). \quad (19)$$

It remains to obtain equations for the transformed potentials V and u . We already know the solutions of Eq. (1) with the unknown potentials $V(x_n)$ and $u(x_n)$ [see formulas (11) and (12)], i.e. eigenvectors of the new Hamiltonian plus associated eigenvalues E_{ν} . As was mentioned, by virtue of the theorem by Gladwell and Willms [13], that is enough for the three-diagonal Hamiltonian matrix (with off-diagonal elements) to be uniquely recovered. These authors used the block Lanczos algorithm. However, we shall apply an outwardly different method pursuing the aim of reproducing final formulas in the continuum limit. Let us multiply both parts of the Schrödinger equation (1) for the solutions $\varphi(x_m, \overset{\circ}{E}_{\mu})$ [Eq. (18)] and $\overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_{\mu})$ by $\overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_{\mu})$ and $\varphi(x_m, \overset{\circ}{E}_{\mu})$, respectively, sum over μ with weight $\overset{\circ}{c}_{\mu}^2$ and subtract from each other the resulting expressions. At fixed m , we perform this procedure for $n = m, m-1, \dots$. In calculating sums (over μ) one should take into account the relation (7). As a result, we get to the following equations for V and u :

$$\begin{aligned} & \{V(x_m) - \overset{\circ}{V}(x_n)\} K(x_m, x_n) + u(x_m) K(x_{m+1}, x_n) - \overset{\circ}{u}(x_n) \\ & \times K(x_m, x_{n+1}) + u(x_{m-1}) K(x_{m-1}, x_n) - \overset{\circ}{u}(x_{n-1}) K(x_m, x_{n-1}) \\ & = \frac{K(x_{m+1}, x_n) - 2K(x_m, x_n) + K(x_{m-1}, x_n)}{\Delta^2} \\ & - \frac{K(x_m, x_{n+1}) - 2K(x_m, x_n) + K(x_m, x_{n-1})}{\Delta^2}, \quad n \leq m-2; \end{aligned} \quad (20)$$

and for $n = m, m-1$

$$\left\{ \begin{aligned} & \frac{u(x_{m-1}) - \overset{\circ}{u}(x_{m-1})}{\Delta} = \frac{K(x_{m+1}, x_{m-1}) - K(x_m, x_{m-2})}{\Delta^2} \\ & - V(x_m) K(x_m, x_{m-1}) + \overset{\circ}{V}(x_{m-1}) K(x_m, x_{m-1}) \\ & - u(x_m) K(x_{m+1}, x_{m-1}) + \overset{\circ}{u}(x_{m-2}) K(x_m, x_{m-2}), \quad n = m-1 \\ & \frac{V(x_m) - \overset{\circ}{V}(x_m)}{\Delta} = \frac{K(x_{m+1}, x_m) - K(x_m, x_{m-1})}{\Delta^2} \\ & - u(x_m) K(x_{m+1}, x_m) + K(x_m, x_{m-1}) \overset{\circ}{u}(x_{m-1}), \quad n = m, \end{aligned} \right. \quad (21)$$

where the terms $K(x_m, x_n)$ for which $m, n > N$ or $m, n < 1$ are omitted. But for $n = m+1$ we obtain that $u(x_m) = \overset{\circ}{u}(x_m)$. There is nothing strange in it because summation is carried out for the term $\overset{\circ}{\varphi}(x_{m+1}, \overset{\circ}{E}_{\mu})$ that is *orthogonal* to all $\overset{\circ}{\varphi}(x_n, \overset{\circ}{E}_{\mu})$, $n < m+1$. But the kernel K containing all the information about new solutions just stands at these summands, see formula (18) for $n = m+1$. In other words, for the case $n = m+1$ the summation expunges everything that bears a relation to the new system

under construction. Indeed, from the Schrödinger equation for $\overset{\circ}{\varphi}(x_{m+1}, \overset{\circ}{E}_\mu)$ multiplied by $\varphi(x_m, \overset{\circ}{E}_\mu)$ and summed over μ with the weight $\overset{\circ}{c}_\mu^2$ we have

$$\sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \overset{\circ}{E}_\mu \varphi(x_m, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_{m+1}, \overset{\circ}{E}_\mu) = \frac{\overset{\circ}{u}(x_m)}{\Delta} - \frac{1}{\Delta^3}.$$

How should one treat this? One variant is that $u(x_m) = \overset{\circ}{u}(x_m)$, which corresponds to restoration of a $m \times m$ submatrix for which the element $\overset{\circ}{u}(x_m)$ is an outer one, see (14). This can also serve as a proof of the (14). Second interpretation is that $u(x_m) \neq \overset{\circ}{u}(x_m)$ (but not for $m = N$), nevertheless (i.e. the above procedure does not work). This takes the place when we have the whole \hat{J} -matrix transformed but the equations (21) by themselves did not allow the computation of $u(x_m)$. How then to uniquely restore the \hat{J} will be discussed a bit later, but now we must ascertain for ourselves how it is possible that the same solution $\varphi(x_m, E)$ may satisfy the Schrödinger equation (1) with *different* potential coefficients $\overset{\circ}{u}(x_m)$ and $u(x_m)$ at x_{m+1} . The matter is that we deal with (finite-difference) non-local potential and this ambiguity is just characteristic of it. Indeed, let $\varphi(x_m, E)$ satisfy the Schrödinger equations with both $\{u_1(x_{m-1}), V_1(x_m), u_1(x_m)\}$ and $\{u_2(x_{m-1}), V_2(x_m), u_2(x_m)\}$. Subtracting these equations from each other we have

$$\begin{aligned} [V_1(x_m) - V_2(x_m)]\varphi(x_m, E) + [u_1(x_m) - u_2(x_m)]\varphi(x_{m+1}, E) \\ + [u_1(x_{m-1}) - u_2(x_{m-1})]\varphi(x_{m-1}, E) = 0. \end{aligned}$$

If we have only one (local) potential coefficient then it would be the same. But now, for several non-local potential coefficients coupling neighbour x -points, that equation clearly demonstrates that $V_1(x_m) - V_2(x_m)$ and others may all be non-zero. Summing up this discussion, we have elucidated that the procedure used for derivation of (20) and (21) cannot distinguish all the variants of $u(x_m)$ -coefficient determination proceeding from a general incapability of giving a unique non-local interaction associated with a certain solution of Schrödinger equation.

However, for the whole vector $\varphi(x_m, E_\nu)$, i.e. the solution defined at *all* the points x_m , $m = 1, \dots, N$ we are able to uniquely derived the quadratic potential matrix \hat{J} whose eigenvectors are $\varphi(x_m, E_\nu)$. Taking $m = N$ we first find $u(x_{N-1})$, $V(x_N)$ and $u(x_N)$. Of course, this requires the knowledge of $u(x_N)$. But we have no more equations for determining the potential coefficient $u(x_N)$. However, we see that the $u(x_N)$ is a continuation of the last N th row of the matrix \hat{J} . This resembles the case with the unfinished restoration of the \hat{J} , see (14), i.e. the potential perturbation (in form of a quadratic matrix) never reached the $u(x_N)$. That is, the $u(x_N)$ is independent of the transformation generated by K -coefficients. But then, taking $\{E_\nu, c_\nu\} = \{\overset{\circ}{E}_\nu, \overset{\circ}{c}_\nu\}$, we see that $K = 0$ and $u(x_N)$ exactly corresponds to the reference potential. Thus, we have $u(x_N) = \overset{\circ}{u}(x_N)$. Next, at the point x_N we have, instead of (21), the following

system of equations

$$\left\{ \begin{array}{l} \frac{u(x_{N-1}) - \overset{\circ}{u}(x_{N-1})}{\Delta} = \frac{\sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \varphi(x_{N+1}, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_{N-1}, \overset{\circ}{E}_\mu) - K(x_N, x_{N-2})}{\Delta^2} \\ -V(x_N)K(x_N, x_{N-1}) + \overset{\circ}{V}(x_{N-1})K(x_N, x_{N-1}) \\ -u(x_N) \sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \varphi(x_{N+1}, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_{N-1}, \overset{\circ}{E}_\mu) + \overset{\circ}{u}(x_{N-2})K(x_N, x_{N-2}), \\ \frac{V(x_N) - \overset{\circ}{V}(x_N)}{\Delta} = \frac{\sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \varphi(x_{N+1}, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_N, \overset{\circ}{E}_\mu) - K(x_N, x_{N-1})}{\Delta^2} \\ -u(x_N) \sum_{\mu=1}^N \overset{\circ}{c}_\mu^2 \varphi(x_{N+1}, \overset{\circ}{E}_\mu) \overset{\circ}{\varphi}(x_N, \overset{\circ}{E}_\mu) + K(x_N, x_{N-1}) \overset{\circ}{u}(x_{N-1}), \end{array} \right. \quad (22)$$

where $\varphi(x_{N+1}, \overset{\circ}{E}_\mu)$ is found from the Schrödinger equation (1):

$$\begin{aligned} \varphi(x_{N+1}, \overset{\circ}{E}_\mu) &= \frac{\Delta^2}{1 - \Delta^2 u(x_N)} [u(x_{N-1}) \varphi(x_{N-1}, \overset{\circ}{E}_\mu) \\ &+ V(x_N) \varphi(x_N, \overset{\circ}{E}_\mu) - \overset{\circ}{E}_\mu \varphi(x_N, \overset{\circ}{E}_\mu)] - \frac{-2\varphi(x_N, \overset{\circ}{E}_\mu) + \varphi(x_{N-1}, \overset{\circ}{E}_\mu)}{1 - \Delta^2 u(x_N)}, \end{aligned} \quad (23)$$

$\varphi(x_n, \overset{\circ}{E}_\mu)$ given by (18) and $u(x_N) = \overset{\circ}{u}(x_N)$. From (22) and (23) we obtain $V(x_N)$ and $u(x_{N-1})$. We then substitute the value $u(x_{N-1})$ (by virtue of the symmetry of potential matrix) into equations (20) for $m = N-1; n = m-2, m-3$ from which we find, in turn, $V(x_{N-1})$ and $u(x_{N-2})$. Afterward, we substitute this last coefficient into equation (20) for $m = N-2; n = m-2, m-3$ and get $V(x_{N-2})$ and $u(x_{N-3})$ and so on. Thus these equations allow the computation of V and u via the solutions of the inverse problem equation (12) – the coefficients $K(x_m, x_n), m > n$ (plus additional requirement concerning $u(x_N)$). For any finite N , these linear equations are uniquely solved. But, with the N large, the numerical instability increases that leads to the well known problem of the ill-posed inversion procedure in the continuum limit. However, let us drop the discussion on that problem here, especially as we only want to reproduce the expression for the continuous potential. So we shall keep on dealing with equations (20) and (21) and next show that passing to the continuum limit in these equations will lead us to the classical results of the Sturm-Liouville inverse problem.

3 Continuum limit

Let us now pass to the limit of the continuous variable x , i.e. to the limit $\Delta \rightarrow 0$ ($N \rightarrow \infty$) so that $\Delta N = \pi N / (N+1) \rightarrow \pi$ in the formulas (11), (12), (20) and (21). Let us recall the standard rules of the transitions from the finite-difference operators to their continuum counterparts:

$$\sum \Delta \rightarrow \int dx; \quad (24)$$

$$\frac{f(x_n) - f(x_{n-1})}{\Delta} \rightarrow \frac{df}{dx}; \quad (25)$$

$$\frac{f(x_{n+1}) - 2f(x_n) + f(x_{n-1}))}{\Delta^2} \rightarrow \frac{d^2f}{dx^2}. \quad (26)$$

Now let us look at the Parseval's relation that takes, in the continuum limit, its usual form for the infinite-dimensional (Hilbert) space

$$\sum_{\mu=1}^{\infty} \overset{\circ}{c}_{\mu}^2 \overset{\circ}{\varphi}(x, \overset{\circ}{E}_{\mu}) \overset{\circ}{\varphi}(y, \overset{\circ}{E}_{\mu}) = \delta(x - y), \quad (27)$$

and the same is for the new regular solutions $\varphi(x, E)$.

In the continuum case, we have $\varphi(0, E) = 0$, $\varphi'(0, E) = 1$. Spectral weight factors are in that case, too, the coefficients of proportionality between normalized eigenfunctions and regular solutions. That is why they are also referred to as norming constants since the multiplication by $c_{\nu} = 1/\int_0^{\pi} \varphi^2(x, E_{\nu})dx$ turns regular solution (at $E = E_{\nu}$) into the normalized one,

$$\Psi'(x, E_{\nu})|_{x=0} = c_{\nu}.$$

The expression for the transformed regular solutions has now the following form (using (24))

$$\varphi(x, E_{\nu}) = \overset{\circ}{\varphi}(x, E_{\nu}) + \int_0^x K(x, y) \overset{\circ}{\varphi}(y, E_{\nu}) dy, \quad (28)$$

and similarly

$$\varphi(x, E) = \overset{\circ}{\varphi}(x, E) + \int_0^x K(x, y) \overset{\circ}{\varphi}(y, E) dy, \quad (29)$$

where $x \in [0, \pi]$. These formulas have just demonstrated that the passage to the limit $\Delta \rightarrow 0$ does exist. For the kernel K of the operator (28) which transforms the solutions to the initial potential into the solutions to the new one (generalized shift operator), we have the continuum analog of Eq. (12) – the inverse problem equation proper:

$$K(x, y) + Q(x, y) + \int_0^x K(x, z)Q(z, y)dz = 0, \quad (30)$$

where the kernel Q is constructed from the unperturbed functions with the old and new spectral parameters [as in Eq. (10)]:

$$Q(x, y) = \sum_{\nu} c_{\nu}^2 \overset{\circ}{\varphi}(x, E_{\nu}) \overset{\circ}{\varphi}(y, E_{\nu}) - \sum_{\mu} \overset{\circ}{c}_{\mu}^2 \overset{\circ}{\varphi}(x, \overset{\circ}{E}_{\mu}) \overset{\circ}{\varphi}(y, \overset{\circ}{E}_{\mu}). \quad (31)$$

For the continuous coordinate, the expression (19) for K has a similar form:

$$K(x, y) = - \sum_{\nu} c_{\nu}^2 \varphi(x, E_{\nu}) \overset{\circ}{\varphi}(y, E_{\nu}) + \sum_{\mu} \overset{\circ}{c}_{\mu}^2 \varphi(x, \overset{\circ}{E}_{\mu}) \overset{\circ}{\varphi}(y, \overset{\circ}{E}_{\mu}). \quad (32)$$

As we have just carried out the passage to the continuum limit in solutions (11), it is clear that such a limit exists for the potential, too. In fact, the expressions for the potential coefficients (20) and (21) were secondary with respect to (11), i.e. we always ‘extract’ them from Schrödinger equation using the information about its solutions (see the section above). As was shown, this procedure is substantially based upon the completeness relation which stands good for *any* Δ , including the continuum case. Moreover, we might use the continuum solution (29) and the Parseval’s relation (27) for the continuum potential to be derived. However, we choose the way of continuum passage in (20) and (21). Another point is that the continuum potential is *local*. Indeed, the u ’s and V in each row of the discrete Sturm-Liouville operator are specified at the very neighbour points x_n and x_{n+1} merging if we pass to the continuum limit, which entails, in turn, superimposing the potential coefficients at one point: $V + 2u$. The distinct feature of the local potential is that the limiting equation must determine it uniquely in contrast to (20) and (21) which, by themselves, could not uniquely specify u ’s and V ’s by reason of a ‘non-local’ character of the interaction in the discrete case (we remember that there was required an additional knowledge of $u(x_N)$ at the boundary of the interval for the uniqueness). But for the Schrödinger equation with a local potential, it is well known that the potential always occur for the unique solution (with given boundary conditions) and vice versa. Thus, we can beforehand anticipate an expression for a unique specification of the local potential in the continuum case.

We shall now prove that, as $\Delta \rightarrow 0$, the equations (20) and (21) go over, respectively, into

$$\begin{aligned} \{V_d(x) - \overset{\circ}{V}_d(y) + 2[u_d(x) - \overset{\circ}{u}_d(y)]\}K(x, y) \\ = \frac{\partial^2}{\partial x^2}K(x, y) - \frac{\partial^2}{\partial y^2}K(x, y), \end{aligned} \quad (33)$$

and

$$\begin{cases} \tilde{V}_d(x) - \overset{\circ}{V}_d(x) + \tilde{u}_d(x) - \overset{\circ}{u}_d(x) = 2\frac{d}{dx}K(x, x) \\ \{\tilde{V}_d(x) - \overset{\circ}{V}_d(x) + \tilde{u}_d(x) - \overset{\circ}{u}_d(x)\}K(x, x) \\ = \frac{\partial^2}{\partial x^2}K(x, y)|_{y=x} - \frac{\partial^2}{\partial y^2}K(x, y)|_{y=x}, \end{cases} \quad (34)$$

where $V_d(x) \equiv \lim_{m \rightarrow \infty} V(x_m)$ and, analogously, $u_d(x) \equiv \lim_{m \rightarrow \infty} u(x_m)$. The tilde sign stands for the potentials obtained in passing to the limit of continuous coordinate in the solutions of Eqs. (21).

In developing these equalities, it is useful to employ the diagonal terms $K(x_n, x_n)$ such as $K(x_{n+1}, x_n) - K(x_n, x_n) \sim O(\Delta)$. First of all, let us consider the term $(K(x_{m+1}, x_{m-1}) - K(x_m, x_{m-2}))/\Delta^2$ in (21). We add to and subtract from the expression in the numerator the term $K(x_{m-1}, x_{m-1}) - K(x_m, x_m)$. Then

$$\begin{aligned} \frac{K(x_{m+1}, x_{m-1}) - K(x_m, x_{m-2})}{\Delta^2} &= \frac{K(x_{m+1}, x_{m-1}) + K(x_{m-1}, x_{m-1})}{\Delta^2} \\ &\quad - \frac{K(x_m, x_m) + K(x_{m-1}, x_{m-1}) - K(x_m, x_m) + K(x_m, x_{m-2})}{\Delta^2} = \zeta. \end{aligned}$$

Next, let us again add to and subtract from the new expression in the numerator the term $2K(x_m, x_{m-1})$:

$$\begin{aligned} \zeta = & \frac{K(x_{m+1}, x_{m-1}) - 2K(x_m, x_{m-1}) + K(x_{m-1}, x_{m-1})}{\Delta^2} \\ & - \frac{K(x_m, x_m) - 2K(x_m, x_{m-1}) + K(x_m, x_{m-2})}{\Delta^2} \\ & + \frac{K(x_m, x_m) - K(x_{m-1}, x_{m-1})}{\Delta^2} \end{aligned}$$

The first two lines in this expression are the second derivatives with respect to the first and second argument of $K(x, y)$ (see (26)). Hence, in the continuum limit they become

$$\frac{\partial^2}{\partial x^2} K(x, y)|_{y=x} - \frac{\partial^2}{\partial y^2} K(x, y)|_{y=x}.$$

The third fraction diverges as $\Delta \rightarrow 0$: $\Delta^{-1} dK(x, x)/dx$. As a result we have

$$\begin{aligned} \frac{K(x_{m+1}, x_{m-1}) - K(x_m, x_{m-2})}{\Delta^2} \longrightarrow & \frac{\partial^2}{\partial x^2} K(x, y)|_{y=x} - \frac{\partial^2}{\partial y^2} K(x, y)|_{y=x} \\ & + \Delta^{-1} \frac{d}{dx} K(x, x). \end{aligned} \quad (35)$$

Likewise, it is not difficult to obtain that

$$\begin{aligned} & \frac{K(x_{m+1}, x_m) - K(x_m, x_{m-1})}{\Delta^2} \\ = & \frac{K(x_{m+1}, x_m) - K(x_m, x_m) + K(x_m, x_m) - K(x_m, x_{m-1})}{\Delta^2} \\ \longrightarrow & \Delta^{-1} \frac{d}{dx} K(x, x), \quad \Delta \rightarrow 0. \end{aligned} \quad (36)$$

In equation (20) we see the finite-difference second derivative in an explicit form. So in continuum case this equation becomes (33). If we introduce $V(x) \equiv V_d(x) + 2u_d(x)$ the term in front of $K(x, y)$ is simply the difference $V(x) - \overset{\circ}{V}(y)$. It is obvious that we introduced a local limiting potential which results from the limiting merging of V-diagonal and nearby u-diagonals.

Now let us multiply both sides of the equations of (21) for $n = m - 1$ and $n = m$ by Δ . We sum the resulting equations and pass to the continuum limit. Then, by virtue of (35) and (36), we get to the first equation in (34) valid to within $O(\Delta)$ (the multiplication by Δ has removed the divergence associated with Δ^{-1}).

The last equation in (34) is not obvious. Indeed, one would think that the term $u(x_m)$ must first be derived from the recurrence procedure (20) and (21) and only afterwards can the passage to the limit $\Delta \rightarrow 0$ be carried out – the procedure of a prodigious complexity. However, we find a way out: we simply take the sum of non-diverging terms (taking into account the expression (35)) in the right-hand side of the

equation (21), $n = m - 1$ to be zero in the continuum limit, i.e., we get the last equation in (34). This by no means contradicts the uniqueness of the sought limiting potential. First, this provides the limiting (for $x = y$) equation for $K(x, y)$ which must exist, obviously. Second, by continuity, the factor $V_d(x) - \overset{\circ}{V}_d(y) + 2[u_d(x) - \overset{\circ}{u}_d(y)]$ in front of $K(x, y)$ must coincide with $\tilde{V}_d(x) - \overset{\circ}{V}_d(x) + \tilde{u}_d(x) - \overset{\circ}{u}_d(x)$ when $x = y$. In other words, that means that $V(x) = V_d(x) + 2u_d(x) = \tilde{V}_d(x) + \tilde{u}_d(x) + \overset{\circ}{u}_d(x)$, i.e. the solutions of (21) go over, in the limit $\Delta \rightarrow 0$, into *the same* local potential $V(x)$, which was beforehand clear. Hence, with the new definition for $V(x)$, we have from (33) and (34):

$$V(x) = \overset{\circ}{V}(x) + 2 \frac{d}{dx} K(x, x). \quad (37)$$

This is the known result of recovering potential in continuum case, which only now became reproducible from a discrete mathematics.

The Eqs. (33) and (34) can now be rewritten as

$$\begin{cases} \{V(x) - \overset{\circ}{V}(y)\} K(x, y) \\ = \frac{\partial^2}{\partial x^2} K(x, y) - \frac{\partial^2}{\partial y^2} K(x, y) \\ V(x) - \overset{\circ}{V}(x) = 2 \frac{d}{dx} K(x, x) \end{cases} \quad (38)$$

This system (added by $K(0, 0) = 0$) represents the classical Goursat problem [for determining $K(x, y)$] and its solvability follows from well known theorems.

The orthogonalization can also be started from the last vector $\varphi(\pi)$ with ‘number’ $x = \pi$ at the right boundary of the interval $[0, \pi]$. Then, instead of solutions $\varphi(x)$, the solutions $f(x)$ will be used such that $f(\pi) = 0$, $f'(\pi) = 1$. The corresponding inverse problem equations, that can be associated with the orthogonalization ‘from the right to the left’, have analogous form as Eqs. (28), (30), (31) and (37), only with other integration limits and different sign in front of the derivative in the expression for $V(x)$:

$$f(x, E) = \overset{\circ}{f}(x, E) + \int_x^\pi K(x, y) \overset{\circ}{f}(y, E) dy; \quad (39)$$

$$K(x, y) + Q(x, y) + \int_x^\pi K(x, z) Q(z, y) dz = 0; \quad (40)$$

$$Q(x, y) = \sum_\nu \gamma_\nu^2 \overset{\circ}{f}(x, E_\nu) \overset{\circ}{f}(y, E_\nu) - \sum_\mu \overset{\circ}{\gamma}_\mu^2 \overset{\circ}{f}(x, \overset{\circ}{E}_\mu) \overset{\circ}{f}(y, \overset{\circ}{E}_\mu); \quad (41)$$

$$V(x) = \overset{\circ}{V}(x) - 2 \frac{d}{dx} K(x, x). \quad (42)$$

Here, the symbol γ_ν stands for the spectral weight factor which is analog of c_ν . The only discrepancy is that the γ_ν characterizes the behaviour of eigenfunction at the right boundary:

$$\Psi(x, E_\nu) = \gamma_\nu f(x, E_\nu), \quad \gamma_\nu = \frac{d}{dx} \Psi(x, E_\nu)|_{x=\pi}.$$

At last, let us mention about the eigenvalue inverse problem for Schrödinger equation added by boundary conditions of arbitrary kind:

$$\Psi'(0) - g\Psi(0) = 0, \quad \Psi'(\pi) + G\Psi(\pi) = 0. \quad (43)$$

Here we also have analogous inversion equations, and as spectral weight factors there appear the values of corresponding eigenfunctions at the interval edges: $c_\nu = \Psi(0, E_\nu)$ or $\gamma_\nu = \Psi(\pi, E_\nu)$.

4 Conclusions

In the present paper we carried out the derivation of main formulas of the inverse eigenvalue problem on the base of its discrete approximation. Several statements of that problem are developed by now, we selected such a statement in which it is possible to reproduce in a maximally straightforward way the future structure of the limiting inversion procedure: the transition from a known system to the system with given spectral data (eigenvalues plus norming constants) but with unknown potential to be recovered. The off-diagonal elements are introduced into the matrix Sturm-Liouville operator (three-diagonal matrix), which is consistent (in contrast to previous works) with the problem statement involving this double set of spectral parameters. At last, in comparison with usual derivation of the continuum inversion equations, our development seems to be none the more complicated. At the same time, the reader acquires the ability to track in more detail additional aspects of the formalism, in particular to look upon the operator transformation realizing the recovering procedure as the orthonormalization of the operator eigenvectors.

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